Perfectoid Fields

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1 Introduction

This talk is the first in the seminar Almost Mathematics, we aim to introduce the concept of perfectoid fields, following the structure of [Bha17]. Basic knowledge of algebra should be sufficient for this talk.

Fontaine and Wintenberger [FW79] proved that

Theorem 1.1. *The absolute Galois groups of $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{F}_p((t))(t^{1/p^\infty})$ are canonically isomorphic.*

The idea of proof: Let K be the completion of $\mathbb{Q}_p(p^{1/p^\infty})$ and let $K^\flat$ be the completion of $\mathbb{F}_p((t))(t^{1/p^\infty})$. It is enough to prove that the absolute Galois group of these two are isomorphic. The choice of notation $K^\flat$ is deliberate since we will later prove that $K^\flat$ is the tilt of K.

In vague terms, the relation between K and $K^\flat$ consists in replacing the prime number p by a formal variable t [Sch11].

The previous theorem generalises to the perfectoid fields:

Theorem 1.2. *When K is perfectoid, the absolute Galois groups of K and $K^\flat$ are canonically isomorphic, where $K^\flat$ is the tilt of K.*

2 Set Up

2.1 Definitions

Definition 1. Let K be a field. An valuation on K is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ satisfying:

1. $|x| = 0$ iff $x = 0$ and $|1| = 1$.
2. $|xy| = |x||y|$ for all $x, y \in K$.
3. $|x + y| \leq |x| + |y|$ for all $x, y \in K$.

A field equipped with an valuation is called a valued field.

Remark 1.

1. The valuation can also be defined to be a map $|\cdot| : K \to \Gamma \cup \{0\}$, where Γ is an ordered abelian multiplicative group. However, for the purpose of our talk, we only consider the case where Γ has rank 1, since the definition of perfectoid field required the valuation to have rank 1. Futher, it can be shown ([AJE05], Proposition 2.1.1) that any rank 1 valuation is equivalent to one which takes value in $\mathbb{R}$ so our definition is equivalent.
2. The term valuation is somewhat unfortunate: if $\Gamma = \mathbb{R}_{>0}$, then the valuation would usually be called a seminorm or absolute value, and the term valuation would be used for (a constant multiple of) the map $x \to -\log|x|$, but we follow Scholze's notation. See [Sch11], Remark 2.3.

Definition 2. An valuation is non-Archimedean (NA) if it satisfies the ultrametric inequality, i.e. $|x+y| \leq \max(|x|, |y|)$.

Example 1. *Over $\mathbb{Q}$, the p-adic valuation $|\cdot|_p$ is non-Archimedean.*

Definition 3 (The valuation topology)**.** If we define a metric $d(x, y) = |x - y|$, then this induces a topology on K.

Definition 4.
$K^\circ = \{x \in K \mid |x| \leq 1\}$ is an open subring of K, called the valuation ring.
$K^{\circ\circ} = \{x \in K \mid |x| < 1\}$ is the maximal ideal of K°.
$K^\circ/K^{\circ\circ}$, usually denoted k, is the residue field.
Any nonzero element $t \in K^{\circ\circ}$ is called a pseudo-uniformizer.

2.2 Basic Results

Lemma 2.1 ("All triangles are isosceles"). *Let $(K, |\cdot|)$ be a non-Archimedean valued field and $x, y \in K$. If $|x| < |y|$, then $|x - y| = |y|$.*

Proof. Because of the ultrametric inequality, $|y| \leq \max(|x|, |x-y|) = |x-y|$ and $|x-y| \leq \max(|x|, |y|) = |y|$. □

Lemma 2.2 (Cauchy sequences in NA fields). *Given a NA valued field K and a sequence $(a_n) \in K$, (a_n) is Cauchy iff $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a_{n+1}| < \epsilon$.*

Proof. If (a_n) is Cauchy, conclusion is clear.

Conversely, suppose that for all $n \geq N$, $|a_n - a_{n+1}| < \epsilon$. Then for all $m > n > N$,

$$|a_m - a_n| \leq \max(|a_i - a_{i+1}|)_{m \geq i \geq n} \leq \epsilon$$

□

Theorem 2.1 ([AJE05], 1.1.4). *There exists a field $\hat{K}$, complete under an valuation $|\cdot|'$ which extends $|\cdot|$ and an embedding $i : K \to \hat{K}$, such that $|x| = |i(x)|'$ for all $x \in K$. The image $i(K)$ is dense in $\hat{K}$.*

If $\hat{K}', i'$ is another such pair, then there exists a unique continuous isomorphism $\phi : \hat{K} \to \hat{K}'$ preserving the valuation and commutes with i.

Lemma 2.3 (Completion of valuation Ring). *Suppose $(K, |\cdot|)$ is a NA valued field, and $(\hat{K}, |\cdot|')$ is its completion. Then $\hat{K}^\circ = \{x \in \hat{K} \mid |x|' \leq 1\}$ is equal to the completion of K° in $\hat{K}$.*

Proof. Suppose that $x_n \to x$, where $x \in \hat{K}^\circ$ and $x_n \in K$. We want to show that $x_n \in K^\circ$.

By Lemma 2.1, eventually $|x_n|' = |x|$, so $|x_n|' \leq 1$ eventually. □

Lemma 2.4. *For a field K with valuation ring K°, and $a, b \in K^\circ$, $a \mid b \iff |a| \geq |b|$.*

Proof. $\implies$ Trivial.

$\impliedby$ Suppose $|a| \geq |b|$, then $|b/a| \leq 1$ so $b/a = c \in K^\circ$. So $ac = b \implies a \mid b$.

In particular, this means that for $a, b \in K^\circ$, either $a \mid b$ or $b \mid a$. □

2.3 Notation

In the following sections we will deal with equivalent classes and elements of $\prod_{n \in \mathbb{N}} R$.

1. Unless otherwise stated, $\prod R$ is equal to $\prod_{n \in \mathbb{N}} R_n$ where $R_n = R$ for all n. Similarly for $\bigsqcup R$.
2. We write an element of $\prod_{n \in \mathbb{N}} R$ as (a_n), with the understanding that the nth element of (a_n) is a_n. We assume the index of (a_n) start from 0.
3. For a ring R, an element $a \in R$, and an equivalence class $\sim$ on R, write $[a]$ as the equivalence class of a.

3 Perfections and Tilting

Definition 5. A characteristic p ring is perfect if the Frobenius map $\phi : x \to x^p$ is bijective; if it is only surjective, we say that is it semiperfect.

Definition 6 (Inverse Limit). Let $(A_n)_{n=1}^\infty$ be a sequence of sets (groups, rings, topological spaces, etc.) equipped with transition maps (homomorphisms, continuous maps etc.) for $j \geq i$ $\phi_{i,j} : A_j \to A_i$, such that $\phi_{ik} = \phi_{ij}\phi_{jk}$ if $i \leq j \leq k$. Then the inverse limit is defined to be:

$$\varprojlim_n A_n = \{(a_n) : a_n \in A_n, \phi_{n+1,n}(a_{n+1}) = a_n\} \subseteq \prod_n A_n$$

Definition 7 (Direct Limit). Let $(A_n)_{n=1}^\infty$ be a sequence of sets (groups, rings, topological spaces, etc.) equipped with transition maps (homomorphisms, continuous maps etc.) for $i \leq j$, $\phi_{i,j} : A_i \to A_j$, such that $\phi_{ik} = \phi_{ij}\phi_{jk}$ if $i \leq j \leq k$. Then the direct limit is defined to be:

$$\varinjlim A_i = \bigsqcup_{i \in \mathbb{N}} A_i / \sim$$

where if $x_i \in A_i, x_j \in A_j$, $x_i \sim x_j$ if there exists $k \geq i, j$ such that $\phi_{jk}(x_j) = \phi_{ik}(x_i)$. Intuitively, two elements in the disjoint union are equivalent if and only if they "eventually become equal" in the direct system.

Remark 2. These two definitions can be generalised to any directed set I, not just the natural number.

Definition 8.

1. Let R be a ring with characteristic p. Then we get a direct system, given by $\{R_n\}_{n=1}^{\infty}$ where $R_n = R$, and $\phi_n : R_n \to R_{n+1}$ sends $x \mapsto x^p$. Set $R_{\text{perf}} := \varinjlim_{\phi} R = \varinjlim_{n} R_n$.

2. Similarly, we get an inverse system $\{R_n\}_{n=1}^{\infty}$ where $R_n = R$, and $\phi_n : R_{n+1} \to R_n$ sends $x \mapsto x^p$. Set $R^{\text{perf}} := \varprojlim_{\phi} R = \varprojlim_{n} R_n$.

3. For any ring R, set $R^\flat := (R/p)^{\text{perf}}$. Unless otherwise specified, this ring is endowed with the inverse limit topology, with each R/p being given the discrete topology.

Remark 3. R^{perf} corresponds to elements in R with p^nth roots for all n, since those are the elements that the inverse limit picks out.

In constructing R_{perf}, we have added $\mathbb{N}$ many copies of R, and in the nth copy, every element is an p^nth root. So the construction of R_{perf} is like adding the p^∞ roots to R.

Both are ways to construct perfect rings, which is what we will prove next.

Lemma 3.1. *When R has characteristic p, both R_{perf} and R^{perf} are perfect. The canonical map $R \to R_{perf}$ (resp. $R^{perf} \to R$) is universal for maps into (resp. from) perfect rings. Moreover, the projection $R^{perf} \to R$ is surjective exactly when R is semiperfect.*

Proof.

Claim 1. *R_{perf} is perfect.*

Injective: suppose that $[a^p] = [b^p]$, for some a, b in the jth and kth copy of R respectively. Then there is some n such that $(a^p)^{p^{n-j}} = (b^p)^{p^{n-k}}$. But that means $a \sim b$.

Surjective: Suppose $a \in \bigsqcup R$ is in the nth copy. Then consider a' in the $n+1$th copy of R, which is equal to a. Then note that $\phi_{n,n+1}(a)$ is equal to a^p in the $n+1$th copy of R, which also happens to be $(a')^p$. Since $\phi_{n,n+1}(a) = (a')^p$, $(a')^p \sim a$, and so a'/p is a pth root of a in R_{perf}. In other words, taking pth power is a right shift map.

Claim 2. *R^{perf} is perfect.*

Injective: Suppose that $(a_n), (b_n) \in \prod R$ is such that $a_n^p = b_n^p$ for all $n \in \mathbb{N}$. But then for all $n > 1$, $a_n^p = a_{n-1} = b_n^p = b_{n-1}$ so $a_n = b_n$ for all n.

Surjective: For $(a_n) \in R^{\text{perf}}$, a_0 has p^nth roots for all n. To get a pth root we simply has to shift the sequence (a_n) to the left. In other words, $(a_{n+1})_{n\in\mathbb{N}}^p = (a_n)_{n\in\mathbb{N}}$.

Claim 3. *Suppose that we have a ring homomorphism $\psi : R \to L$, where L is a perfect field. Then we can find a ring homomorphism $\Psi : R_{perf} \to L$, which commutes with ψ and the canonical map $i : R \to R_{perf}$.*

For any $\alpha \in R_{\text{perf}}$, take a representative a in the nth copy of R, such that $[a] = \alpha$. Let $\Psi(\alpha) = \phi^{-n}(\psi(a))$, which exists and is unique because L is perfect.

We now prove that this map is well defined: indeed, $\Psi([\phi_{n,n+1}(a)]) = \Psi([a])$.

To see that the map commutes: for all $r \in R$, $i(r)$ is equal to $[r]$ where r is in the first copy of R, so $\Psi([r]) = \psi(r)$.

Claim 4. *Suppose that we have a ring homomorphism $\psi : L \to R$, where L is a perfect field. Then we can find a ring homomorphism $\Psi : L \to R^{perf}$, which commutes with ψ and the canonical map $i : R^{perf} \to \mathbb{R}$ given by the projection to the first element.*

For any $a \in L$, $\psi(a) \in R$ has a p^nth root for all $n \in \mathbb{N}$. This is because a has a p^nth root and ψ is a ring homomorphism.

$\psi(a)$ now determined an element in R^{perf}: let $c_0 \in R$ be equal to $\psi(a)$; let c_{n+1} be the pth root of c_n. Then, $(c_n) \in R^{\text{perf}}$.

Let $\Psi(a) = (c_n) \in R^{\text{perf}}$. This commutes with ψ and i because $c_0 = \psi(a)$.

Claim 5. *Canonical map $i : R^{perf} \to R$ is surjective iff R is semiperfect.*

The projection map is surjective means every element in R has a pth root. Conversely if R is semiperfect then every $r \in R$ gives rise to an element in R^{perf}. □

Now let us see a few examples.

Example 2.

1. $\mathbb{F}_p[t]_{perf} = \mathbb{F}_p[t^{1/p^\infty}]$ *and* $\mathbb{F}_p[t]^{perf} = \mathbb{F}_p$.
2. $\mathbb{Z}_p^\flat = \mathbb{F}_p$, $\mathbb{F}_p[t]^\flat \cong \mathbb{F}_p$.

Proof.

1. $\mathbb{F}_p[t]_{\text{perf}} = \varinjlim_\phi \mathbb{F}_p[t]$.

   We define a map $\psi : \bigsqcup \mathbb{F}_p[t] \to \mathbb{F}_p[t^{1/p^\infty}]$: if $a \in \bigsqcup \mathbb{F}_p[t]$ is in the nth copy, and that $a = \sum_{i=0}^m \gamma_i t^i \in \mathbb{F}_p[t]$, then we define a map $\psi(a) = \sum_{i=0}^m \gamma_i t^{i/p^n}$. Note that $\psi(\phi_{n,n+1}(a)) = \sum_{i=0}^m \gamma_i t^{ip/p^{n+1}} = \psi(a)$.

   Now we define $\Psi : (\bigsqcup \mathbb{F}_p[t]/\sim) \to \mathbb{F}_p[t^{1/p^\infty}]$: for any $\alpha \in (\bigsqcup \mathbb{F}_p[t]/\sim)$, take a representative $a \in \mathbb{F}_p[t]$, and let $\Psi(\alpha) = \psi(a)$. We need to show that it is well defined.

   But suppose that we have a in the jth copy, and b in the kth copy are such that and that $a \sim b$. Then there is a $n > j, k$ such that $\phi_{j,n}(a) = \phi_{k,n}(b)$.

   Then $\psi(a) = \psi(\phi_{j,n}(a)) = \psi(\phi_{k,n}(a)) = \psi(b)$.

   Now we know Ψ is well defined. Ψ is injective because $\psi(a) = 0 \iff a = 0$. It is surjective because for any element α in $\mathbb{F}_p[p^{1/p^\infty}]$, $\alpha^{p^n} \in \mathbb{F}_p[t]$ for n large enough. So by raising α to a high enough pth power, we can find an element with image α.

   $\mathbb{F}_p[t]^{\text{perf}} = \varprojlim_\phi \mathbb{F}_p[t]$.

   Take any $a \in \varprojlim_\phi \mathbb{F}_p[t] \in \prod_{n\in\mathbb{N}} \mathbb{F}_p[t]$. Just project it onto the first element.

   Because the first element has infinitely many p^nth root for any $n \in \mathbb{N}$, the first element must be in $\mathbb{F}_p$, and the first element determined a. So the map is injective and surjective.

2. $\mathbb{Z}_p/p = \mathbb{F}_p$, which is perfect, so $\mathbb{F}_p^{\text{perf}} = \mathbb{F}_p$. $\mathbb{F}_p[t]/p = \mathbb{F}_p[t]$, so done by the previous part.

□

Lemma 3.2. *Let $f : R \to S$ be a map of characteristic p rings that is surjective with nilpotent kernel. Then $R^{perf} \cong S^{perf}$ and $R_{perf} \cong S_{perf}$.*

Proof. Suppose $a^e = 0$ for all $a \in \ker(f)$, and let k be big enough such that $p^k > e$.
There is a natural map $(a_n) \in R^{\text{perf}} \mapsto (f(a_n)) \in S^{\text{perf}}$. It is

1. well defined because f is a ring homomorphism and R has characteristic p;
2. injective because suppose $(f(a_n)) = 0$, then $a_n \in \ker(f)$ for all n. Further for all n, $a_n = a_{n+k}^{p^k} = 0$ because $p^k > e$. So $(a_n) \in R^{\text{perf}} = 0$ and the map is injective.
3. surjective: take $(b_n) \in S^{\text{perf}}$, and for all $n \in \mathbb{N}$, let $a_n \in R \coloneqq B_{n+k}^{p^k}$, where $f(B_{n+k}) = b_{n+k}$. We need to show that the choice of a_n is well-defined and independent of the choice of B_{n+k}. Suppose $f(C_{n+k}) = f(B_{n+k}) = b_{n+k}$, then $(C_{n+k} - B_{n+k})^{p^k} = 0$. Hence $C_{n+k}^{p^k} = B_{n+k}^{p^k}$. Further, $f(a_n) = b_{n+k}^{p^k} = b_n$.

   Now that's left to do is to check that (a_n) is indeed an element of the inverse limit. Indeed, $a_{n+1}^p = B_{n+k+1}^{p^{k+1}}$. But $f(B_{n+k+1}^p) = b_{n+k+1}^p = b_{n+k}$, so $(B_{n+k+1}^p)^{p^k} = B_{n+k}^{p^k} = a_n$.

Similarly, there is a natural map $\psi : \bigsqcup R \to \bigsqcup S$ sending a in the nth copy of R to $f(a)$ in the nth copy of S. This induces a map $[a] \in R_{\text{perf}} \mapsto [\psi(a)] \in S_{\text{perf}}$. It is

1. well defined because if $a \sim b$ then $f(a) \sim f(b)$ since f is a ring homomorphism and preserves powers.
2. injective because suppose that $f([a]) = [f(a)] = 0 \in R_{\text{perf}}$, then $f(a)^{p^n} = f(a^{p^n}) = 0$ for some n, but that means a^{p^n} is in the nilpotent kernel, and so $a^{p^m} = 0$ for m large enough, and $[a] = [0]$.
3. surjective because for any $[b] \in S_{\text{perf}}$, where $b \in \bigsqcup S$ is in the nth copy of S, there exists a in the nth copy of R such that $f(a) = b$.

□

We will next show an elementary lemma:

Lemma 3.3. *Let R be a ring (doesn't have to have characteristic p), and let $t \in R$ be an element such that $p \in (t)$. Given $a, b \in R$ with $a = b \pmod t$, we have $a^{p^n} = b^{p^n} \pmod{t^{n+1}}$ for all n.*

Proof. We will prove this by induction. If $n = 0$, there is nothing to prove.

Now, suppose we know $a^{p^n} = b^{p^n} \pmod{t^{n+1}}$ we want to prove that $a^{p^{n+1}} = b^{p^{n+1}} \pmod{t^{n+2}}$.

Suppose that $a^{p^n} = b^{p^n} + t^{n+1}c$ for some $c \in R$, then raising both sides by a power of p, we see that

$$a^{p^{n+1}} = b^{p^{n+1}} + p \cdot t^{n+1} \cdot d + t^{p(n+1)}c^p$$

for some $d \in R$. Then because $p \in (t)$, we have our result. □

We will use this elementary result to prove something very important for future applications, since it gives a "strict" description of the tilt functor.

Lemma 3.4. *Assume R is p-adically complete. The projection map $R \to R/p$ induces a bijection $\varprojlim_\phi R \to \varprojlim_\phi R/p = R^\flat$ of multiplicative monoids.*

Proof. **Injective:** Suppose that $(a_n), (b_n) \in \varprojlim_\phi R$ are sequences such that $(a_n/p) = (b_n/p) \in \varprojlim_\phi R/p$.

Then we know that $a_n = b_n \pmod{p}$ for all n. From the definition of inverse limit, we know that $a_{n+k}^{p^k} = a_n$ for all n, k. Similarly for b_n.

Fix n, k. Because we know $a_{n+k} = b_{n+k} \pmod{p}$, from the previous lemma and the fact we just mentioned, we have $a_n = b_n \pmod{p^{k+1}}$.

This holds for all k. Now, because R is p-adically complete, we must have $a_n = b_n$ for all n.

Surjective: Fix $(\alpha_n) \in \varprojlim_\phi R/p$. For all $\alpha_n \in R/p$, pick any lift a_n, then we will modify this so that it is an element of the inverse limit.

Because $\forall n, k, \alpha_{n+k+1}^p = \alpha_{n+k}$, we have that $a_{n+k+1}^p = a_{n+k} \pmod{p}$ for all n, k. Hence from the previous lemma, $a_{n+k+1}^{p^{k+1}} = a_{n+k}^{p^k} \pmod{p^{k+1}}$. Now, this holds for all k, therefore the sequence $k \mapsto a_{n+k}^{p^k}$ is a Cauchy sequence. Because R is p-adically complete, there is a limit $a_{n+k}^{p^k} \to b_n$ for some $b_n \in R$.

Now we claim $b_n/p = \alpha_n$. $a_{n+k}^{p^k} = a_n \pmod{p}$, therefore $a_{n+k}^{p^k}/p = \alpha_n$ for all k, so $b_n/p = \alpha_n$ too.

Now we check that $b_{n+1}^p = b_n$: $b_{n+1}^p = \lim_{k\to\infty} a_{n+1+k}^{p^{k+1}} = b_n$ by defintion. So $(b_n) \in \varprojlim_\phi R$, and its image under the projection is (α_n). □

Remark 4. Note that the construction of (b_n) from (a_n) in the second half of the proof above is well-defined (i.e., independent of auxiliary choices of the lifts), and gives an explicit inverse to the projection $\varprojlim_\phi R \to \varprojlim_\phi R/p = R^\flat$.

Definition 9 (Sharp Map)**.** In Lemma 3.4, via the projection to the first term, we get a multiplicative map $\sharp : R^\flat \to R$ denoted $f \mapsto f^\sharp$.

Remark 5.

1. The image under the sharp map is exactly those $f \in R$ that admit a compatible system of p-power roots. We shall sometimes call such items *perfect*.
2. The sharp map is not additive. However, it is additive mod p: $(x+y)^\sharp = x^\sharp + y^\sharp \pmod{p}$. This is because $(x+y)^p \equiv x^p + y^p \pmod{p}$.

   Therefore the bijection $\varprojlim_\phi R \cong \varprojlim_\phi R/p$ is additive $\pmod{p}$, hence so is $\sharp$.

Using the $\sharp$ map, we can understand valuation rings under tilting; this will be useful when discussing adic spaces later.

Now we see a series of results, that show that tilt preserves a lot of properties of R.

Lemma 3.5. *If a p-adically complete ring R is a domain then so is the tilt $R^\flat$.*

Proof. From Lemma 3.4, we know that We have $R^\flat \cong R$ as a multiplicative monoid.

Fix elements $(a_n), (b_n) \in \varprojlim_\phi R$ with $a_n \cdot b_n = 0$ for all n. Then either a_0 or b_0 vanishes as R is a domain.

Without the loss of generality, assume $a_0 = 0$. Then, since transition map involves the raising of the powers, and R is a domain, $a_n = 0$ for all n, and so $(a_n) = 0$. □

Lemma 3.6. *If a p-adically complete domain R is a valuation ring (of Frac(R)), the same is true for its tilt $R^\flat$. In fact, if $|\cdot| : R \to \mathbb{R} \cup \{0\}$ is an valuation on R, then the map $R^\flat \xrightarrow{\sharp} R \xrightarrow{|\cdot|} \mathbb{R}$ gives the corresponding valuation on $R^\flat$.*

Proof. Suppose that there is an valuation $|\cdot|$ on $\mathrm{Frac}(R)$, such that R is its valuation ring.

Claim: For any $a=(a_n), b=(b_n) \in R^\flat$, $a \mid b \iff a^\sharp \mid b^\sharp$

Proof: From the bijection $\mathbb{R}^\flat \cong \varprojlim_\phi R$, get $A=(A_n), B=(B_n) \in \varprojlim_\phi R$ corresponding to a and b. It satisfies $A \mid B \iff a \mid b$.

Now, because R is a valuation ring, we have that $A_0 \mid B_0$ or vice versa. Wlog assume that $A_0 \mid B_0$, then, $A_n^{p^n} | B_n^{p^n}$ for all n, but this tells us that $|A_n| \geq |B_n|$ for all n and so $A_n \mid B_n$ for all $n \geq 1$. But also, $(A_{n+1}/B_{n+1})^p = (A_n/B_n)$, thus $(A_n) \mid (B_n)$ in $\varprojlim_\phi R$. Conversely $(A_n) \mid (B_n) \implies A_0 \mid B_0$.

So now

$$(a_n) \mid (b_n) \iff (A_n) \mid (B_n) \iff A_0 \mid B_0 \iff a^\sharp \mid b^\sharp$$

Once we have the claim, take any $x/y \in \mathrm{Frac}(R^\flat)$ (with $x, y \in R^\flat$), send it to $x^\sharp/y^\sharp \in \mathrm{Frac}(R)$.

Claim 6. *$|x/y|^\flat = |x^\sharp/y^\sharp|$ is a valuation.*

Proof. The only axiom that is difficult is the ultrametric inequality, since we do not know that $\sharp$ map is additive.

We want to prove that $|(x+y)^\sharp| \leq \max(|x^\sharp|, |y^\sharp|)$ for $x, y \in K^\flat$.

Because R is a valuation ring in $\mathrm{Frac}(R)$, we have that $x^\sharp \mid y^\sharp$ or vice versa. WLOG assume $x^\sharp \mid y^\sharp$, but this is equivalent to $x \mid y$ from the previous claim. Suppose $z \in R^\flat$ satisfy $y = xz$.

Now by dividing both sides, it suffices to show that $|(1+z)^\sharp| \leq 1 + |z^\sharp|$. However, $(1+z) \in R^\flat$, so $(1+z)^\sharp \in R$, and so $|(1+z)^\sharp| \leq 1$.

Alternatively, note that the sharp map is additive mod p: $(x+y)^\sharp = x^\sharp + y^\sharp \pmod p$. Hence $(x+y)^\sharp = x^\sharp + y^\sharp \pmod p$ in R, and so $|(x+y)^{p^n\sharp}| \leq \max(|x^{p^n\sharp}|, |y^{p^n\sharp}|, |p|)$. Now if we take p^n roots, and let $n \to \infty$, we see that $|(x+y)^\sharp| \leq \max(|x^\sharp|, |y^\sharp|)$.

We now show that under this valuation, $R^\flat$ is its valuation ring.

Suppose that $|x^\sharp/y^\sharp| \leq 1$, then $|x^\sharp| \leq |y^\sharp|$ which by Lemma 2.4 means $y^\sharp \mid x^\sharp$ in R. But this is equivalent to $y \mid x$ in $R^\flat$, so $x/y \in R^\flat$. Conversely, if $x \in R^\flat$, $x^\sharp \in R$ ahs valuation ≤ 1. □

4 Perfectoid Fields

Next, we introduce perfectoid fields. These are NA fields that contain "lots of" p-power roots [Bha17]. The main result is that the tilt of (the ring of integers of) a perfectoid field K is a perfectoid field K of characteristic p that reflects the algebraic properties of K.

Definition 10. Fix a prime number p. A perfectoid field K is a complete NA field with residue characteristic p such that:

1. The value group $|K^\times| \subseteq \mathbb{R}_{>0}$ is not discrete.
2. K°/p is semiperfect, i.e. the Frobenius map $K^\circ/p \to K^\circ/p$ is surjective.

Example 3.

1. *$\mathbb{Q}_p$ is not perfectoid.*
2. *Completely valued field of residue characteristic p where every element $x \in K$ has a pth root. This condition is satisfied, for example, if K is algebraically closed. An example would be $K = \mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$.*

Proof.

1. $\mathbb{Q}_p$ has discrete valuation, so not perfectoid.
2. Because every element has a pth root, in particular the value group cannot be discrete. But also the Frobenius map $K^\circ \to K^\circ$ is already surjective, so K°/p is semiperfect.

□

Example 4. $K = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$.

Proof. The valuation ring of $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ is $\widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$. This is because $\mathbb{Q}_p(p^{1/p^\infty}) = \bigcup_{n\in\mathbb{N}} \mathbb{Q}_p(p^{1/p^n})$. Each $\mathbb{Q}_p(p^{1/p^n})$ has valuation ring $\mathbb{Z}_p[p^{1/p^n}]$, hence the valuation ring of $\mathbb{Q}_p(p^{1/p^\infty})$ is $\mathbb{Z}_p[p^{1/p^\infty}]$. Finish by Lemma 2.3. Now we have

$$\frac{\widehat{\mathbb{Z}_p[p^{1/p^\infty}]}}{p} = \frac{\mathbb{Z}_p[p^{1/p^\infty}]}{p} = \frac{\mathbb{Z}_p[t^{1/p^\infty}]/(t-p)}{p} = \frac{\mathbb{F}_p[t^{1/p^\infty}]}{t-p}$$

The first equality is true due to density: every $x \in \mathbb{Z}_p[\widehat{p^{1/p^\infty}}]$ satisfy $|x-y| < |p|$ for some $y \in \mathbb{Z}_p[p^{1/p^\infty}]$, hence $x = y \pmod p$.

The second equality is true, because we can build a map: $\phi : \mathbb{Z}_p[t^{1/p^\infty}] \to \mathbb{Z}_p[p^{1/p^\infty}]$, given by $t \mapsto p$. Define $\Phi : \mathbb{Z}_p[t^{1/p^\infty}]/(t-p) \to \mathbb{Z}_p[p^{1/p^\infty}]$ given by $[a] \mapsto \phi(a)$. Can prove this is well defined and an isomorphism.

But now since $\mathbb{F}_p[t^{1/p^\infty}]$ is perfect we get semi-perfectness, and that K is a perfectoid field.

In fact the tilt of this field is $\mathbb{F}_p(\widehat{(t^{1/p^\infty}}))$, which is intuitively what we will get by replacing all of the occurances of p with t.

This is because in the next lecture, we should see that the ring of integers of $K^\flat$ can be constructed as $K^{\circ\flat}$, and so the tilt is

$$\varprojlim_{\phi} \frac{\mathbb{F}_p[t^{1/p^\infty}]}{t-p} = \mathbb{F}_p(\widehat{(t^{1/p^\infty}}))$$

□

Lemma 4.1. *Let $(K, |\cdot|)$ be a perfectoid field.*

1. *In $|K^\times|$ every element is a pth root.*
2. *We have $(K^{\circ\circ})^2 = K^{\circ\circ}$.*
3. *The ring K° is not Noetherian.*

Proof.

1. We temporarily call $x \in K^\circ$ small if $|p| < |x| \leq 1$.

   Claim: $|x|$ is p divisible if $x \in K^\times$ is small.

   Proof: from the second property from the perfectoid definition, we have $y, z \in K^\circ$ such that $x = y^p + pz$. Now, $|x| \leq \max(|y|^p, |p||z|)$. But $|p||z| \leq |p| < |x|$, so $|x| = |y|^p$, hence p divisible.

   Now, we generalise to any x. Because $|K|$ is not discrete, the containment $|p|^{\mathbb{Z}} \subseteq |K|$ is a strict containment. pick any $x \in K^\circ$ such that $|x| \notin |p|^{\mathbb{Z}}$, then there exists $n \in Z$ such that $|p|^n < |x| < |p|^{n-1}$. Then $y \coloneqq x/p^{n-1}$ is small.

   Because $|p| < |y|$, so by Lemma 2.4, $y \mid p$, and $\exists w \in K^\circ$: $p = yw$. But then w is also small, hence $|p|$ is a pth power. Hence every element is a pth power.

2. From the previous point, we see that for every $x \in K^{\circ\circ}$, $|x| = |y^p|$ for some $y \in K^{\circ\circ}$, but then $x = y^p u$ for some unit $u \in K^{\circ\circ}$, hence in $K^{\circ\circ 2}$.

3. $K^{\circ\circ}$ is not finitely generated: if it is, by some elements $\{e_1, \ldots, e_n\}$, then every element in $= K^{\circ\circ}$ has its valuation bounded above by some $1 - \epsilon$ for some $\epsilon > 0$. But that's impossible since there are elements of arbitrarily small valuation.

□

Remark 6. The proof above shows that $|K^*| \subseteq \mathbb{R}_{>0}$ is generated by $\{|x| \mid |p| < |x| < 1\}$. This observation will be useful later in analyzing the value group under tilting.

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